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ON THE CLASSIFICATION OF THE ELEMENTS OF A RING

by

RUBY USHER HIGHTOWER, M.A.

Submitted in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy

in the

GRADUATE SCHOOL

of the

UNIVERSITY OF MISSOURI

1927

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INTRODUCTION

This study investigates a method of classifying the elements of an abstract ring. The method secures, incidentally, certain properties usually chosen as the basis for classification, such as congruence in a group with respect to a self-conjugate subgroup and congruence with respect to an ideal modulus.

In "Congruences" [1] Masazo Sono has investigated the properties of ideals in an abstract ring; in particular, the conditions for unique factorization of such ideals. Congruence is defined by him as a change of definition of equality. In this paper congruence is defined by classification of the elements of a ring. The study deals with algebras having the commutative property; but because of frequent reference to the works of M. Sono, his terminology is retained.

Dedekind, in the study of algebraic numbers, introduced the idea of ring ideals. He showed that for ideals prime to a certain ideal, called the conductor of a ring, the same laws hold as in the ring composed of all the algebraic integers of a field. A similar problem is studied in this paper. It is shown that for all ideals prime to a certain ideal C_0 the laws established by M. Sono for ideals of a ring R are extended to ideals of a subring $\overline{R}$.

Elements of R are abstract. They have two laws of composition, called addition and multiplication, which are subject to the commutative, associative, and distributive laws. The null element is designated by 0. The unit element u, such that $ru = r$ for each element r of R, is designated by 1, $u + u$ by 2, etc. These numerals may be treated as ordinary integers.[2]

The property of an ideal A in R that $AR = A$ is called the invariance of A in R; the property $A + R = R$ is called the "law of absorption." Two ideals A and B of R are said to be relatively prime if $A + B = R$.

The subring $\overline{R}$ is called a proper subring if $\overline{R}$ is less than R.

[1] Sono, M., *Congruences I, II, III.*

[2] Sono, M., *Congruences II*, page 114.

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Sect. 1. A Classification of the Elements of the Ring R

Assume that the ring R has the abstract elements $r_0, r_1, r_2, \cdots$, finite or infinite in number, and that these elements can be separated into non-overlapping classes $C_0, C_1, C_2, \cdots$. Denote the elements of C_i by $r_{i1}, r_{i2}, \cdots$. From any two classes C_i and C_j select the elements r_{ik} and r_{jl}, and form the sum $r_{ik} + r_{jl}$. If for every k and l these sums belong to the same class C_s, the notation is

$$C_s = C_i + C_j.$$

If all possible products of the form $r_{ik} \times r_{jl}$ belong to the same class C_t, the notation is

$$C_t = C_i \times C_j.$$

We shall first assume that the classification has been made so that for any two classes C_i and C_j all possible sums $r_{ik} + r_{jl}$ belong to the same class. The set is closed with respect to addition.

Since R is a ring, it contains the zero element 0; let C_0 be the class to which 0 belongs. Then if r_{0k} is any element of C_0 and r_{jl} any element of C_j, $r_{0k} + r_{jl}$ belongs to the class determined by the sum of $0 + r_{jl}$, i.e., to the class C_j. It follows that

$$C_0 + C_j = C_j.$$

Hence C_0 is the identical element in the set of classes. Moreover, since addition in R is associative and commutative, addition of classes is associative and commutative. Finally, since R contains the negative of each of its elements, if r_{jl} belongs to C_j, and $(-r_{jl})$ to C_j', it follows that $C_j + C_j'$ contains 0, and

$$C_j + C_j' = C_0.$$

Therefore each class has an inverse. Hence the classes form an Abelian group with addition as the composition.

It will next be shown that the class C_0 is a module.

Since $C_0 + C_j = C_j$, $C_0 + C_0 = C_0$, or the sum of any two elements of C_0 is contained in C_0. To show that the difference of any two elements of C_0 is contained in C_0, we first show that

if C_i contains r_{ik}, and C_i' contains $(-r_{ik})$, the negative of each element of C_i is in C_i'. If r_{il} is any other element of C_i, r_{il} when added to its negative $(-r_{il})$ determines C_0. But because of the property used as the basis for classification, r_{il} when added to $(-r_{ik})$ determines C_0. It follows that $(-r_{ik})$ and $(-r_{il})$ belong to the same class C_i'. Let C_i' be denoted by $(-C_i)$. Since C_0 contains r_{0k} and $(-C_0)$ contains $(-r_{0k})$, $(-C_0) + C_0$ contains 0. Hence C_0 is a module.

The class $C_i - C_i = C_0$, since it contains 0. Hence

$$r_{ik} \equiv r_{il} \quad \text{modulo } C_0.$$

It follows that the classification of the elements of R as above effected, with the classes constituting an Abelian group with respect to addition, is a separation of the elements of R into its residue classes modulo C_0, where C_0 is a module.

Since R is an Abelian group with respect to addition, while C_0 is a subset of R closed with respect to the law of composition of the group, C_0 is a subgroup of R. Hence the classification of R above effected is an example of congruence in a group with respect to a self-conjugate subgroup as a modulus.[3]

Let us now add to the group of classes $C_0, C_1, \cdots$, which is closed with respect to addition, the condition that this set is also closed with respect to multiplication. Since the associative, commutative and distributive laws hold for the multiplication of elements, they hold for the multiplication of classes; and the group of classes C becomes a ring of classes.

Assume the existence of a subring $\overline{C}$ of C. This subring will contain C_0, which is the zero element of C. Consider S, the set of elements contained in $\overline{C}$. Since the classes of $\overline{C}$ constitute a ring, the elements of S will also constitute a ring, although the converse is not necessarily true. Then S is a subring of R; it will be called $\overline{R}$.

It will now be shown that C_0 is an ideal in R and in $\overline{R}$. Since the ring of classes C is closed with respect to multiplication, the product of any element r_{ik} of C_i by the element 0 of C_0 determines C_0 as the product of the two classes C_0 and C_i. It follows that the product of any element of R by an element of C_0 is contained in C_0, in consequence of the laws assumed for the determination

[3] Sono, M., *Congruences I*, p. 206. Miller, Blickfeldt, and Dickson, *Finite Groups*, Part I, p. 67.

of classes. Hence C_0 is an ideal of R. Since C_0 is an element of the subring of classes $\overline{C}$, C_0 is an ideal of $\overline{R}$ also.

If C_0 is the greatest common divisor of ideals common to R and $\overline{R}$, C_0 is defined as the conductor or $\overline{R}$ relative to R; otherwise C_0 is contained in the conductor. For C_0 to be coextensive with the conductor, C_0 must be a maximal ideal of R which is also an ideal of $\overline{R}$, i.e., there can be no other ideal of R contained in $\overline{R}$ and containing C_0.

The following theorem will now be proved:

The necessary and sufficient condition that the ideal C_0 be maximal is that $\overline{C}$, the ring of classes of $\overline{R}$ modulo C_0, contain no proper subring C' that is invariant in C.

To prove that the condition stated in the theorem is sufficient, first assume that there does exist in $\overline{R}$ another ideal I of R such that $C_0 < I$. It will now be shown that $\overline{C}$ contains a proper subring C', invariant in C and composed of the classes of I modulo C_0.[4]

C_0 is invariant in I. Each class of I modulo C_0 is contained in a unique class of $\overline{C}$. It follows that the classes of $\overline{C}$ containing the elements of I modulo C_0 form a subring C' of $\overline{C}$, owing to the property used as the basis for classification. Moreover, C' is invariant in $\overline{C}$. For if $\overline{r}$ is any element of $\overline{R}$, and i any element of I, $\overline{r}i$ is an element of I. With respect to the corresponding classes, it is evident that the class containing $\overline{r}$ is an element of $\overline{C}$, and that the classes containing i and $\overline{r}i$ are elements of C'. Hence C' is invariant in $\overline{C}$. Similarly, C' is invariant in C. Therefore C' is an ideal of C that is in $\overline{C}$. It follows that if $\overline{C}$ contains no proper subring C' invariant in C, C_0 is a maximal ideal of R contained in $\overline{R}$.

To prove that the condition stated in the theorem is necessary, it will first be shown that if C_0 is an ideal of R and $\overline{R}$, and C' a subring of $\overline{C}$ that is invariant in C, $\overline{R}$ contains an ideal $I < \overline{R}$ that is invariant in R, such that $C_0 < I$, and C' is the set of classes of I modulo C_0. To prove this, consider those elements of $\overline{R}$ which belong to classes of $\overline{C}$ giving elements of C'. These elements constitute a subring I of $\overline{R}$. Since $C' < \overline{C}$, $I < \overline{R}$. Now the class C_0 is determined by 0; hence $C_0 < I$. If the class determined by $\overline{r}$ is an element of $\overline{C}$, and the class determined by i an element of C', the invariance of C' in C shows that the

[4] Cf. Dickson, L. E., *Algebras and Their Arithmetics*, pp. 41–42.

class determined by $\bar{r}i$ is an element of C'. It follows that if i is an element of I, and $\bar{r}$ an element of $\bar{R}$, $\bar{r}i$ is an element of I; or I is invariant in $\bar{R}$. Similarly, it may be shown that I is invariant in R. Or I is an ideal of R and $\bar{R}$; and the classes of I modulo C_0 are the elements of C'. From the theorem just proved it follows that if C_0 is a maximal ideal of R which is an ideal of $\bar{R}$, the ring of classes $\bar{C}$ contains no subring of classes C' invariant in C. For if $\bar{C}$ contained an ideal C' invariant in C, $\bar{R}$ would contain an ideal $I > C_0$ and invariant in R, contrary to the assumption that C_0 is maximal.

Hence the necessary and sufficient condition that C_0 be the conductor of $\bar{R}$ relative to R is that the subring $\bar{C}$ contain no proper subring C' invariant in C. A sufficient but not necessary condition is that $\bar{C}$ be simple; i.e., that $\bar{C}$ contain no other subring than the zero element C_0.

Since $\bar{R}$ is any subring of R, the theorem may be extended to a composition-series of rings. Assume a series of rings R, R_1, R_2, $\cdots$, R_n, such that each R_{i+1} is contained in R_i and $R_{i+1} < R_i$. The necessary and sufficient condition that the ideal C_0 be the conductor of each R_m relative to R_k $[k < m]$ is that each ring of classes of R_m modulo C_0 contain no subring invariant in the ring of classes of R_k modulo C_0.

It is obvious that the properties assumed as the basis for classification secure uniqueness of composition.[5]

[5] Sono, M., *Congruences I*, pp. 204–206.

Sect. 2. Regular Ideals in R

In Congruences II, Masazo Sono has derived the laws for the factorization of an ideal of finite norm in a proper ring. In Congruences III, he has proved that the results relating to resolution hold true when the norms of the ideals are infinite, and that the conditions for unique factorization are substantially the same in this case.

In this paper, the theorems established by M. Sono for ideals in a ring R are extended to those ideals in the subring $\bar{R}$ which are prime to C_0. Such ideals are said to be regular. Their characteristic properties are:

$$[A] \qquad \bar{R}I = I,$$

$$[B] \qquad I + C_0 = \bar{R}.^{6}$$

Theorem I. If I is a regular ideal in $\bar{R}$, $\bar{R}$ is the ring I^0 of I, where I^0 is used to denote the totality of all elements of R such that $II^0 = I$. Since $\bar{R}I = I$, $\bar{R} \leqq I^0$, and since $1 \leqq I^0$, while $RC_0 = C_0$, it follows that $C_0 \leqq C_0I^0 = C_0$. Hence

$$C_0 = C_0I^0.$$

Moreover, since $I^0I = I$, and $I + C_0 = \bar{R}$,

$$\bar{R} = I + C_0 = II^0 + C_0I^0 = \bar{R}I^0.$$

Also, since $1 \leqq \bar{R}$, $I^0 \leqq \bar{R}I^0 = \bar{R}$. Hence $I^0 = \bar{R}$.

Theorem II. If I' and I'' are any two regular ideals in $\bar{R}$, their greatest common divisor $I' + I''$, their cross-cut $I' \wedge I''$, and their product $I'I''$ are also regular ideals in $\bar{R}$.

$$(a) \qquad \bar{R}(I' + I'') = \bar{R}I' + \bar{R}I'' = I' + I''. \qquad [A]$$

By the associative law and the law of absorption,

$$(I' + I'') + C_0 = I' + (I'' + C_0) = I' + \bar{R} = \bar{R}. \qquad [B]$$

(b) The product of any element of $\bar{R}$ by an element common to two ideals gives an element contained in each ideal. Or

$$\bar{R}(I'' \wedge I') = I'' \wedge I'. \qquad [A]$$

[6] Bachmann: *Allgemeine Arithmetik der Zahlenkorper*, Chapter 9.

Since $\bar{R} = I' + C_0$, $\bar{R}I'' = I'I'' + C_0I''$. Since $I'I'' \leqq RI' = I'$, and $I'I'' \leqq \bar{R}I'' = I''$, while $I''C_0 \leqq \bar{R}C_0 = C_0$, it follows that

$$I'' \leqq (I' \wedge I'') + C_0,$$

and hence

$$\bar{R} = I'' + C_0 \leqq (I' \wedge I'') + C_0.$$

Conversely, it is obvious that $I' \wedge I'' + C_0 \leqq \bar{R}$. Hence

$$(I' \wedge I'') + C_0 = \bar{R}. \qquad [B]$$

(*c*) $$\bar{R}(I'I'') = (\bar{R}I')I'' = I'I''. \qquad [A]$$

Also, since $\bar{R} = I' + C_0$ and $\bar{R} = I'' + C_0$,

$$\bar{R}\bar{R} = I'I'' + I'C_0 + I''C_0 + C_0C_0.$$

But since $\bar{R}\bar{R} = \bar{R}$, while $I'C_0$, $I''C_0$ and C_0C_0 are all contained in C_0, it follows that

$$\bar{R} \leqq I'I'' + C_0.$$

Conversely,

$$I'I'' + C_0 \leqq \bar{R}.$$

Hence

$$I'I'' + C_0 = R. \qquad [B]$$

Corollary: The product $I = I'I''$ is contained in each of the ideals I' and I''.

Sect. 3. Correspondence between Regular Ideals in $\bar{R}$ and Ideals in R Prime to C_0

Theorem I. If I is a regular ideal in $\bar{R}$, RI is an ideal in R prime to C_0.

(1) $$R(RI) = (RR)I = RI. \qquad [A]$$

Since $I + C_0 = \bar{R}$,

(2) $$RI + RC_0 = R\bar{R}.$$

(3) Since $\bar{R} \leqq R$, $R\bar{R} \leq RR = R$. Conversely, since $\bar{R}$ contains 1,

(4) $$R \leqq R\bar{R}; \qquad \text{and} \qquad R\bar{R} = R.$$

Substituting (4) in (2):

(5) $$RI + C_0 = R. \qquad [B]$$

Theorem II. I is the cross-cut of RI and $\bar{R}$.

Let a be any element of $(RI \wedge \bar{R})$, i and c elements of I and C_0 respectively. Since $I + C_0 = \bar{R}$, it follows that a as an element of $\bar{R}$ may be written: $a = i + c$, for properly chosen i and c. But $c = a - i \leqq RI$, since a and i are both contained in RI. Hence $c \leqq (RI \wedge C_0)$. Therefore, from $a = i + c$,

(1) $$(RI \wedge \bar{R}) \leqq I + (RI \wedge C_0).$$

Conversely, when i is an element of I, and c is contained in both RI and C_0, each element $i + c$ is an element in RI and in $\bar{R}$. Hence

(2) $$I + (RI \wedge C_0) \leqq (RI \wedge \bar{R}),$$

and

(3) $$(RI \wedge \bar{R}) = I + (RI \wedge C_0).$$

It is proved by M. Sono that if two ideals are relatively prime, their product is equal to their cross-cut.[7] Hence

(4) $$(RI \wedge C_0) = (RI)C_0 = (RC_0)I = IC_0.$$

Substituting (4) in (3):

(5) $$(RI \wedge \bar{R}) = I + IC_0 = I.$$

[7] Sono, M., *Congruences II*, p. 117.

THEOREM III. R is the greatest common divisor of RI and $\bar{R}$.

$$R = RI + C_0 \leqq RI + \bar{R} \leqq R.$$

Hence

$$RI + \bar{R} = R.$$

THEOREM IV. If J is an ideal in R relatively prime to C_0, and if I is the cross-cut of J and R, I is a regular ideal of $\bar{R}$.

Let i be an element of $(J \wedge \bar{R})$, $\bar{r}$ an element of $\bar{R}$. Since $\bar{R} \leqq R$, $\bar{r}i$ is contained in J and also in $\bar{R}$. Then $\bar{r}i$ is an element of I. Moreover, since $\bar{R}$ contains the unit element,

$$I \leqq \bar{R}I, \qquad \text{and hence} \qquad \bar{R}I = I. \qquad [A]$$

If j, c, and r are elements of J, C_0, and R respectively, since $J + C_0 = R$, each element of R may be written:

$$r = j + c.$$

As a special case, each element $\bar{r}$ of $\bar{R}$ may be written

$$\bar{r} = j^{(i)} + c^{(i)}.$$

Hence

$$j^{(i)} = \bar{r} - c^{(i)}.$$

It follows that $j^{(i)}$ is an element of $\bar{R}$, and hence of $J \wedge \bar{R}$. Therefore,

$$\bar{R} \leqq J \wedge \bar{R} + C_0 \leqq \bar{R}.$$

Hence

$$I + C_0 = \bar{R}. \qquad [B]$$

Corollary: RI is an ideal in R relatively prime to C_0 and identical with J. Since $I \leqq J$, $RI \leqq RJ = J$. Since $RI + C_0 = R$, $RI \cdot J + C_0 \cdot J = R \cdot J = J$. From $C_0 J = J$ and $C_0 J = C_0 \leqq \bar{R}$,

$$C_0 J \leqq J \wedge \bar{R} = I \leqq RI.$$

Hence $J \leqq RI$. Therefore $J = RI$.

The correspondence between regular ideals in $\bar{R}$ and ideals in R prime to C_0 is now established. To each regular ideal I of $\bar{R}$ there corresponds an ideal J of R prime to C_0, and conversely.

$$J = RI; \qquad I = J \wedge \bar{R}; \qquad \text{and} \qquad (J + \bar{R}) = R.$$

The norm of I under $\bar{R}$ is defined as the number of classes of elements of $\bar{R}$ modulo I; it is defined as the order of the quotient ring $\bar{R}/I$, and is written: $N^{(\bar{R})}(I)$.

THEOREM V. The group of classes of R modulo J is simply isomorphic with the group of classes of $\bar{R}$ modulo I, when $I + C_0 = \bar{R}$, and $J = RI$.

M. Sono has proved the following theorem: [8]

If each element of a group H is permutable with a group K, the groups $(H + K)/K$ and H/L are simply isomorphic, where L is the subgroup common to H and K.

Let us apply this theorem to the rings R and $\bar{R}$, and the ideals I and $J = RI$. Each of these aggregates is an Abelian group with respect to addition, while each element of RI is permutable with the elements of R. It follows that $(RI + \bar{R})/RI$, or R/J, is simply isomorphic with $(\bar{R}/(RI \wedge \bar{R}))$, or $\bar{R}/I$. If the order of each quotient group is finite, it follows that the norm of J under R is equal to the norm of I under $\bar{R}$.

THEOREM VI. If I and I' are two ideals in $\bar{R}$ such that $RI' = RI''$, $I' = I''$. For

$$I' = \bar{R} \wedge RI' \qquad \text{and} \qquad I'' = \bar{R} \wedge RI''.$$

THEOREM VII. If $J = J'J''$ and if J and J' are both relatively prime to C_0, J'' is also relatively prime to C_0.

$$J' + C_0 = R, \qquad \text{and} \qquad J + C_0 = R.$$

Since $J'' + C_0$ is an ideal in R,

$$\begin{aligned} J'' + C_0 &= R(J'' + C_0) = (J' + C_0)(J'' + C_0) \\ &= J + J'' \cdot C_0 + C_0(J' + C_0) \\ &= J + J''C_0 + C_0R \\ &= J + C_0 + J''C_0 \\ &= R + J''C_0 = R. \end{aligned}$$

Corollary: There are ideals I, I' and I'' in $\bar{R}$ such that $J = RI$, $J' = RI'$, and $J'' = RI''$.

THEOREM VIII. If an ideal J in R is divisible by an ideal J' in R, it follows with respect to corresponding ideals in $\bar{R}$ that I is divisible by I', and conversely. Since $J = J'J''$, $RI = RI' \cdot RI'' = R \cdot I'I''$. Hence, from theorem VI, $I = I'I''$.

Conversely, if an ideal I in $\bar{R}$ is divisible by another ideal I' in $\bar{R}$, a third ideal I'' in $\bar{R}$ exists such that $I = I'I''$. If I and I' are relatively prime to C_0, I'' is also prime to C_0. The proof

[8] Sono, M., *Congruences II*, p. 208.

is the same as that for theorem VII. Hence there are in R ideals $J = RI$, $J' = RI'$, and $J'' = RI''$, prime to C_0. But from $I = I'I''$ it follows that

$$RI = R(I'I'') = RR(I'I'') = (RI')(RI'').$$

Hence

$$J = J'J''.$$

THEOREM IX. If I' and I, two regular ideals in $\bar{R}$, are prime to each other, the corresponding ideals J' and J of R are also prime to each other, and conversely. Since $I + I' = \bar{R}$, $R(I + I') = RI + RI' = R\bar{R} = R$. Hence

$$J + J' = R.$$

Conversely, if $J + J' = R$, $R(I + I') = R = R\bar{R}$. Now $\bar{R}$ is itself an ideal of R, while $(I + I')$, by Sect. 2, theorem II, is also an ideal of R. Hence, from Sect. 3, theorem VI, $I + I' = \bar{R}$.

It is proved by M. Sono [9] that if two ideals in R are relatively prime and if their norms under R are both finite, the norm of their product is equal to the product of their norms.

THEOREM X. If two regular ideals in $\bar{R}$ are prime to each other, and if their norms under $\bar{R}$ are finite, the norm of their product is equal to the product of their norms.

Let I and I' be two ideals in $\bar{R}$ prime to each other and each prime to C_0. Then, by Sect. 3, theorem IX, the corresponding ideals J and J' are prime to each other. From Sect. 3, theorem V,

$$\begin{aligned} N^{(R)}(J) &= N^{(\bar{R})}(I), \\ N^{(R)}(J') &= N^{(\bar{R})}(J'), \\ N^{(R)}(JJ') &= N^{(\bar{R})}(II'). \end{aligned}$$

But by the theorem of M. Sono quoted above

$$N^{(R)}(JJ') = N^{(R)}(J) \cdot N^{(R)}(J').$$

Hence

$$N^{(\bar{R})}(II') = N^{(\bar{R})}(I) \cdot N^{(\bar{R})}(I').$$

In consequence of the complete correspondence between regular ideals in $\bar{R}$, and ideals of R prime to C_0, the laws established by M. Sono for the factorization of ideals in R can be extended to ideals in $\bar{R}$. The results of Sono's investigation of the factorization of ideals in R are as follows:

[9] Sono, M., *Congruences II*, p. 117.

Let $R, A_1, A_2, \cdots A_n$ $(n > 2)$ be the chief composition-series of a ring R, with a last term A_n. If any one of the quotient rings $A_1/A_2, A_2/A_3, \cdots A_{n-1}/A_n$ be a field, the ideal A_n may be expressed as the product of two ideals prime to each other. The converse theorem also holds. Both theorems may be summed up as follows:

An ideal of a proper ring can or can not be resolved into two factors prime to each other according as it does or does not contain a power of a maximal ideal.

In order that every composite ideal of a ring R may be resolved into factors, and that the factorization shall be unique, it is necessary and sufficient that for every maximal regular ideal P of R, and for every exponent e, the equation should hold:

$$N^{(R)}(P^e) = [N^{(R)}(P)]^e.$$[10]

[10] Sono, M., *Congruences II*, pp. 132, 136, 148.

Sect. 4. An Infinite Abelian Group with a Finite Base

Elements were assumed to be abstract. But it will now be shown that if R, considered an infinite Abelian group with respect to addition, has a finite base, and if $\bar{R}$ is a subgroup of R of finite index, $\bar{R}$ is a ring defined by an algebraic equation.

In an Abelian group G of degree h $(h > 1)$, with basal elements $B_1, B_2, \cdots B_r$, of degrees $h_1, h_2, \cdots h_r$ respectively $(h_i > 1)$, each element G is obtained uniquely in the form: $C = B_1^{x_1} \cdot B_2^{x_2} \cdots B_r^{x_r}$, where the integers x_i independently of one another run over complete residual systems modulo h_i.

When the infinite Abelian group G has a finite base $B_1, B_2, \cdots B_r$, each subgroup U of G of finite index has a base $U_1, U_2, \cdots U_n$ of the form:

$$\begin{aligned} U_1 &= B_1^{r_{11}} \cdot B_2^{r_{12}} \cdots B_n^{r_{1n}} \\ U_2 &= \qquad B_2^{r_{22}} \cdots B_n^{r_{1n}} \\ &\cdot \\ &\cdot \\ &\cdot \\ U_n &= \qquad\qquad B_n^{r_{nn}}, \end{aligned}$$

with $r_{ii} \neq 0$ for $i = 1, 2, \cdots n$. The index of U under G is

$$j = |r_{11}\, r_{22} \cdots r_{nn}|.^{11}$$

If G has addition as the composition, and if α is any element of U, we have the system of equations:

$$\begin{aligned} \alpha U_1 &= a_{11}U_1 + \cdots a_{1n}U_n \\ \alpha U_2 &= a_{21}U_1 + \cdots a_{2n}U_n \\ &\cdot \\ &\cdot \\ &\cdot \\ \alpha U_n &= a_{n1}U_1 + \cdots a_{nn}U_n, \end{aligned}$$

where each a_{ij} is an integer as defined in the introduction.

[11] Hecke, E., *Algebraischen Zahlen*, p. 39.

Hence

$$\begin{vmatrix} (a_{11} - \alpha), & a_{12}, & \cdots & a_{1n} \\ a_{21}, & (a_{22} - \alpha), & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1}, & a_{n2}, & \cdots & (a_{nn} - \alpha) \end{vmatrix} = 0.$$

Therefore α is an algebraic number.

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BIOGRAPHY

The author of this dissertation was born near Covington, Georgia, on June 17, 1880. She was graduated from the High School at Jackson, Georgia, and from Shorter College of Rome, Georgia. After further study at the University of Chicago and the University of London, she entered the University of Georgia, where she received the degree of Master of Arts in 1919. In the summer session of 1921 she entered the Graduate School of the University of Missouri, where she spent one year and five summer sessions in resident study. She was appointed Honorary Fellow in Mathematics for the session of 1924–25. In the spring of 1927 she returned for her oral examinations in completion of the requirements for the degree of Doctor of Philosophy.

Her teaching experience has been received in the High School at Dublin, Georgia; Alabama Normal College; Central College of Conway, Arkansas; Anderson College of Anderson, South Carolina; Hardin College of Mexico, Missouri; and Shorter College of Rome, Georgia. Since 1919 she has been head of the department of mathematics in Shorter College.